J Eng Math (2008) 61:99–109
DOI 10.1007/s10665-007-9187-x

Thermal stresses in a viscoelastic trimaterial with a combination of a point heat source and a point heat sink

G. A. Porter · C. T. Chuang · C. K. Chao · R. C. Chang

Received: 18 September 2006 / Accepted: 8 August 2007 / Published online: 7 September 2007

Abstract A general solution for a thermoviscoelastic trimaterial combined with a point heat source and a point heat sink is presented in this work. Based on the method of analytic continuation associated with the alternation technique, the solutions to the heat-conduction and thermoelastic problems for three dissimilar, sandwiched media are derived. A rapidly convergent series solution for both the temperature and stress field, expressed in terms of an explicit general term of the corresponding homogeneous potential, is obtained in an elegant form. The hereditary integral in conjunction with the Kelvin–Maxwell model is applied to simulate the thermoviscoelastic properties, while a thermorheologically simple material is considered. Based on the correspondence principle, the Laplace transformed thermoviscoelastic solution is directly determined from the corresponding thermoelastic one. The real-time solution can then be solved numerically by taking the inverse Laplace transform. A typical example concerning the interfacial stresses generated from a combined arrangement of a heat source and sink are discussed in detail. The corresponding thin-film problem is also discussed.

Keywords Analytic continuation · Interfacial stresses · Thermoviscoelasticity · Trimaterial

1 Introduction

Stresses induced by the thermal mis-match of dissimilar media have been an important topic since temperature differences are rapidly increasing in modern electron devices. Thermal stresses become the main criterion to cause multilayered media failure. Moreover, with the advance in material science, many multilayered components are polymer-based making them particularly sensitive to the temperature change. The interaction between time and temperature in multilayered viscoelastic structures has important applications because in recent years these components are widely used in numerous engineering designs.

The analysis of the multilayered problem is complicated since the solutions must be forced to satisfy the continuity conditions of multiple interfaces. Consequently, the conventional approach to stress analysis for multilayer-media

G. A. Porter · C. T. Chuang · C. K. Chao (✉)
Department of Mechanical Engineering, National Taiwan University of Science and Technology, 43 Keelung Rd., Taipei Taiwan, ROC
e-mail: ckchao@mail.ntust.edu.tw

R. C. Chang
Department of Mechanical Engineering, St. John's University, 499 Tam King Road, Sec. 4 Tamsui, Taipei County Taiwan, ROC

problems requires to solve a system of simultaneous equations with a large number of unknown constants. For example, Iyengar and Alwar [1], as well as Chen [2], solved the semi-infinite medium composed of isotropic layers. Some more efficient methods dealing with the complicated continuity conditions have been proposed, and are mentioned below. Based on the transfer-matrix approach, which is expressed in terms of infinite series expansions allowing solutions with various orders of approximation to be obtained, Bufler [3] solved the elasticity problem in a multilayered medium. Based on the Fourier-transform technique in association with the stiffness-matrix approach, Choi and Thangjitham [4,5] obtained solutions for multilayered anisotropic elastic media. Choi and Earmme [6,7] employed the alternating technique to obtain solutions for singularity problems in an isotropic and anisotropic trimaterial. Chao and Chen [8] used analytic continuation combined with the alternating technique to solve the thermoelastic problem of an isotropic trimaterial.

For the thermoviscoelastic analysis, several authors [9,10] derived the appropriate forms of free energy and the corresponding stress–strain relations and dissipation energy for a thermorheologically simple material from the viewpoint of irreversible thermodynamics. Because of the complexity of these problems, most results reported in the literature are found in numerical approximation [11–13].

In this work, we consider the problem of a viscoelastic trimaterial combined with a point heat source and a point heat sink. The term, "trimaterial", as defined here, represents an infinite solid composed of three dissimilar materials bonded along two parallel interfaces. Based on the method of analytical continuation in conjunction with the alternate technique, the trimaterial solution can be derived in a series form from the corresponding homogeneous solution. A variety of problems (such as the bimaterial problem, or the thin layer bonded to a half plane, or the finite strip of thin film, etc.), can be treated as special cases of the present study.

The general solution of the temperature field T, the total heat flux Q, the displacement derivatives and stresses for a thermoelastic medium are

$$T = \frac{1}{2}\left[\theta(z) + \bar{\theta}(\bar{z})\right], \tag{1}$$

$$Q = \frac{-k}{2}\left[\theta(z) - \bar{\theta}(\bar{z})\right], \tag{2}$$

$$2G(u_1' + u_2') = \kappa\Phi(z) - \bar{\Omega}(\bar{z}) - (z - \bar{z})\bar{\Phi}'(\bar{z}) + 2G\beta\theta(z), \tag{3}$$

$$\sigma_2 - \mathrm{i}\sigma_{12} = \Phi(z) + \bar{\Omega}(\bar{z}) + (z - \bar{z})\bar{\Phi}'(\bar{z}), \tag{4}$$

where $z = x_1 + \mathrm{i}x_2$ denotes the complex coordinate, and $\theta(z)$ is a complex temperature function. A bar over the variable denotes the conjugate of a complex, number or variable, and a prime indicates the derivative with respect to its argument; k and G denote the heat conductivity and elastic shear modulus, respectively. Furthermore, the material constants κ and β are defined as $\kappa = 3-4\nu$ and $\beta = (1+\nu)\alpha$ for plane strain, and $\kappa = (3-\nu)/(1+\nu)$ and $\beta = \alpha$ for plane stress, where ν is Poisson's ratio, and α is the coefficient of thermal expansion. The components of displacements and stresses can be expressed in terms of two complex stress functions, $\Phi(z)$ and $\Omega(z)$, associated with a temperature function, $\theta(z)$, when the thermal effect is considered.

2 Temperature field of a trimaterial

Referring to Fig. 1, consider a dissimilar triple-layer medium with two perfectly bonded interfaces $L(x_2 = 0)$ and $L^*(x_2 = h)$. Suppose that the regions $D_a(x_2 > h)$, $D_b(h > x_2 > 0)$, and $D_c(x_2 < 0)$ are occupied by materials a, b, and c, respectively.

For the first argument, we regard regions D_a and D_b to be composed of the same material b and region D_c of material c. If $\theta_0(z)$ signifies a potential for a singularity in an infinite homogeneous plane of material b, then $\theta_{c1}(z)$ (analytic in D_c) and $\theta_1(z)$ (analytic in $D_a \cup D_b$) are introduced to satisfy the continuity conditions across L as

$$\theta(z) = \begin{cases} \theta_0(z) + \theta_1(z), & z \in D_a \cup D_b \\ \theta_{c1}(z), & z \in D_c \end{cases}. \tag{5}$$

The continuity of temperature and total heat flux across the interface, L, requires

$$\begin{aligned}&\theta_0(x_1)+\theta_1(x_1)+\bar{\theta}_0(x_1)+\bar{\theta}_1(x_1)=\theta_{c1}(x_1)+\bar{\theta}_{c1}(x_1),\\ &k_b[\theta_0(x_1)+\theta_1(x_1)]-k_b[\bar{\theta}_0(x_1)+\bar{\theta}_1(x_1)]=k_c\theta_{c1}(x_1)-k_c\bar{\theta}_{c1}(x_1).\end{aligned}\tag{6}$$

Through standard analytic-continuation arguments it follows that

$$\theta_1(z)+\bar{\theta}_0(z)=\bar{\theta}_{c1}(z),\quad z\in D_a\cup D_b,\qquad \bar{\theta}_1(z)+\theta_0(z)=\theta_{c1}(z),\quad z\in D_c\tag{7}$$

and

$$k_b\theta_1(z)-k_b\bar{\theta}_0(z)=-k_c\bar{\theta}_{c1}(z),\quad z\in D_a\cup D_b,\qquad -k_b\bar{\theta}_1(z)+k_b\theta_0(z)=k_c\theta_{c1}(z),\quad z\in D_c.\tag{8}$$

Uncoupling Eqs. 7 and 8, we obtain

$$\theta_1(z)=\Lambda_{cb}\bar{\theta}_0(z),\quad z\in D_a\cup D_b,\qquad \theta_{c1}(z)=\Pi_{cb}\theta_0(z),\quad z\in D_c\tag{9}$$

with

$$\Lambda_{cb}=\frac{k_b-k_c}{k_b+k_c},\quad \Pi_{cb}=\frac{2k_b}{k_c+k_b}.\tag{10}$$

Since this result is based on the assumption that region D_a is made up of material b, it cannot satisfy the continuity conditions at the interface L^*, which lies between material a and b.

For the second argument, we assume regions D_b and D_c are made up of the same material b, and region D_a is composed of material a. Additional terms, $\theta_{b1}(z)$ (analytic in $D_b\cup D_c$) and $\theta_{a1}(z)$ (analytic in D_a), are introduced to satisfy the continuity conditions across the interface L^* such that

$$\theta(z)=\begin{cases}\theta_{a1}(z), & z\in D_a\\ \theta_0(z)+\theta_1(z)+\theta_{b1}(z), & z\in D_b\cup D_c\end{cases}.\tag{11}$$

Similarly, the continuity of the temperature and the total heat flux across the interface L^* requires

$$\begin{aligned}&\theta_0(x_1+\mathrm{i}h)+\theta_1(x_1+\mathrm{i}h)+\theta_{b1}(x_1+\mathrm{i}h)+\bar{\theta}_0(x_1-\mathrm{i}h)+\bar{\theta}_1(x_1-\mathrm{i}h)+\bar{\theta}_{b1}(x_1-\mathrm{i}h)\\ &\quad=\theta_{a1}(x_1+\mathrm{i}h)+\bar{\theta}_{a1}(x_1-\mathrm{i}h)\end{aligned}$$

and

$$\begin{aligned}&k_b[\theta_0(x_1+\mathrm{i}h)+\theta_1(x_1+\mathrm{i}h)+\theta_{b1}(x_1+\mathrm{i}h)]-k_b[\bar{\theta}_0(x_1-\mathrm{i}h)+\bar{\theta}_1(x_1-\mathrm{i}h)+\bar{\theta}_{b1}(x_1-\mathrm{i}h)]\\ &\quad=k_a\theta_{a1}(x_1+\mathrm{i}h)-k_a\bar{\theta}_{a1}(x_1-\mathrm{i}h).\end{aligned}\tag{12}$$

According to analytic continuation, this leads to the results,

$$\begin{aligned}&\theta_0(z+\mathrm{i}h)+\theta_1(z+\mathrm{i}h)+\bar{\theta}_{b1}(z-\mathrm{i}h)=\theta_{a1}(z+\mathrm{i}h),\quad z\in D_a,\\ &\bar{\theta}_0(z-\mathrm{i}h)+\bar{\theta}_1(z-\mathrm{i}h)+\theta_{b1}(z+\mathrm{i}h)=\bar{\theta}_{a1}(z-\mathrm{i}h),\quad z\in D_b\cup D_c\end{aligned}\tag{13}$$

and

$$\begin{aligned}&k_b\theta_0(z+\mathrm{i}h)+k_b\theta_1(z+\mathrm{i}h)-k_b\bar{\theta}_{b1}(z-\mathrm{i}h)\quad=k_a\theta_{a1}(z+\mathrm{i}h),\quad z\in D_a,\\ &-k_b\bar{\theta}_0(z-\mathrm{i}h)-k_b\bar{\theta}_1(z-\mathrm{i}h)+k_b\theta_{b1}(z+\mathrm{i}h)=-k_a\bar{\theta}_{a1}(z-\mathrm{i}h),\quad z\in D_b\cup D_c.\end{aligned}\tag{14}$$

Uncoupling Eqs. 13 and 14, we obtain

$$\theta_{a1}(z)=\Pi_{ab}[\theta_0(z)+\theta_1(z)],\quad z\in D_a,\qquad \theta_{b1}(z)=\Lambda_{ab}[\bar{\theta}_0(z-2ih)+\bar{\theta}_1(z-2\mathrm{i}h)],\quad z\in D_b\cup D_c\tag{15}$$

with coefficients,

$$\Pi_{ab}=\frac{2k_b}{k_a+k_b},\quad \Lambda_{ab}=\frac{k_b-k_a}{k_b+k_a}.\tag{16}$$

Since this result is based on the assumption that region D_c is made up of the same material b, it cannot satisfy the continuity conditions at the interface L.

For the third argument, we again assume D_a and D_b are made up of the same material b, and region D_c is composed of material c. Additional terms, $\theta_2(z)$ (analytic in $D_a \cup D_b$) and $\theta_{c2}(z)$ (analytic in D_c) are introduced to satisfy the continuity conditions across the interface L. By using a way similar to the previous approach, we can let

$$\theta(z) = \begin{cases} \theta_{b1}(z) + \theta_2(z), & z \in D_a \cup D_b \\ \theta_{c2}(z), & z \in D_c \end{cases} \tag{17}$$

and thereby obtain

$$\begin{aligned} \theta_2(z) &= \Lambda_{cb}\bar{\theta}_{b1}(z), \quad z \in D_a \cup D_b \\ \theta_{c2}(z) &= \Pi_{cb}\theta_{b1}(z), \quad z \in D_c. \end{aligned} \tag{18}$$

The method of analytic continuation is repeatedly performed across the two interfaces to achieve the additional terms; $\theta_{ai}(z)$, $\theta_{bi}(z)$, $\theta_{ci}(z)$ and $\theta_i(z)$, for $i = 2, 3, 4 \ldots$. Consequently, we find the complete solution of $\theta(z)$ as

$$\theta(z) = \begin{cases} \theta_a(z), & z \in D_a \\ \theta_0(z) + \theta_{ba}(z) + \theta_{bc}(z), & z \in D_b \\ \theta_c(z), & z \in D_c \end{cases} \tag{19}$$

with

$$\begin{aligned} \theta_a(z) &= \sum_{i=1}^{n} \theta_{ai}(z) = \Pi_{ab}\theta_0(z) + \Pi_{ab}\sum_{i=1}^{n}\theta_i(z), \\ \theta_{ba}(z) &= \sum_{i=1}^{n} \theta_{bi}(z) = \Lambda_{ab}\bar{\theta}_0(z-2\mathrm{i}h) + \Lambda_{ab}\sum_{i=1}^{n}\bar{\theta}_i(z-2\mathrm{i}h), \\ \theta_{bc}(z) &= \sum_{i=1}^{n}\theta_i(z), \\ \theta_c(z) &= \sum_{i=1}^{n}\theta_{ci}(z) = \Pi_{cb}\theta_0(z) + \Pi_{cb}\sum_{i=1}^{n}\theta_{bi}(z) = \Pi_{cb}\theta_0(z) + \Pi_{cb}\Lambda_{ab}\bar{\theta}_0(z-2\mathrm{i}h) \\ &\quad + \Pi_{cb}\Lambda_{ab}\sum_{i=1}^{n}\bar{\theta}_i(z-2\mathrm{i}h), \end{aligned} \tag{20}$$

where

$$\theta_i(z) = \begin{cases} \Lambda_{cbcb}\bar{\theta}_0(z), & i = 1 \\ \Lambda_{cb}\Lambda_{ab}[\theta_0(z-2\mathrm{i}h) + \theta_1(z+2\mathrm{i}h)], & i = 2 \\ \Lambda_{cb}\Lambda_{ab}\theta_{i-1}(z+2\mathrm{i}h), & i \geq 3 \end{cases} \tag{21}$$

For a point heat source of intensity q_0 located at the point z_s, and a point heat sink of the same intensity located at the point z_k, the potential of the corresponding homogeneous problem is

$$\theta_0(z) = \frac{-q_o}{2\pi k_b}\log\left(\frac{z-z_s}{z-z_k}\right). \tag{22}$$

Note that Eq. 19 represents the solution when the singularities are located in region D_b. For the singularities located in other regions, the solution can also be found by using the same procedure.

3 Stress field of a trimaterial

Consider the stress field of a dissimilar triple-layer medium, with singularities located in the middle layer; see Fig. 1. Similar to the previous approach, we first regard regions D_a and D_b as being composed of the same material b, and region D_c made of material c. Let

$$\Phi(z) = \begin{cases} \Phi_0(z) + \Phi_1(z), & z \in D_a \cup D_b \\ \Phi_{c1}(z), & z \in D_c \end{cases},$$

$$\Omega(z) = \begin{cases} \Omega_0(z) + \Omega_1(z), & z \in D_a \cup D_b \\ \Omega_{c1}(z), & z \in D_c \end{cases}, \tag{23}$$

where $\Phi_0(z)$ and $\Omega_0(z)$ are the corresponding homogeneous solutions, while $\Phi_1(z)$, $\Phi_{c1}(z)$, $\Omega_1(z)$, and $\Omega_{c1}(z)$ are analytic functions. The continuity of traction and displacement across L yields

$$\begin{aligned} &\Phi_{c1}(x) + \bar{\Omega}_{c1}(x) = \Phi_0(x) + \Phi_1(x) + \bar{\Omega}_0(x) + \bar{\Omega}_1(x), \\ &\frac{1}{2G_c}[\kappa_c\Phi_{c1}(x) - \bar{\Omega}_{c1}(x)] + \beta_c\theta_c(x) = \frac{1}{2G_b}[\kappa_b\Phi_0(x) + \kappa_b\Phi_1(x) - \bar{\Omega}_0(x) - \bar{\Omega}_1(x)] \\ &\qquad + \beta_b[\theta_0(x) + \theta_{ba}(x) + \theta_{bc}(x)]. \end{aligned} \tag{24}$$

By standard analytic continuation arguments, it follows

$$\begin{aligned} &\bar{\Omega}_{c1}(z) = \Phi_1(z) + \bar{\Omega}_0(z), \quad z \in D_a \cup D_b, \\ &\Phi_{c1}(z) = \Phi_0(z) + \bar{\Omega}_1(z), \quad z \in D_c \end{aligned} \tag{25}$$

and

$$\begin{aligned} -\frac{\bar{\Omega}_{c1}(z)}{2G_c} &= \frac{1}{2G_b}[\kappa_b\Phi_1(z) - \bar{\Omega}_0(z)] + \beta_b\theta_{ba}(z), \quad z \in D_a \cup D_b, \\ \frac{\kappa_c\Phi_{c1}(z)}{2G_c} + \beta_c\theta_c(z) &= \frac{1}{2G_b}[\kappa_b\Phi_0(z) - \bar{\Omega}_1(z)] + \beta_b[\theta_0(z) + \theta_{bc}(z)], \quad z \in D_c. \end{aligned} \tag{26}$$

From Eqs. 25 and 26 we can find

$$\begin{aligned} &\Phi_1(z) = V_{bc}\bar{\Omega}_0(z) + \theta_{1bc}(z), \quad \Omega_{c1}(z) = (V_{bc}+1)\Omega_0(z) + \bar{\theta}_{1bc}(z), \quad \Omega_1(z) = U_{bc}\bar{\Phi}_0(z) + \bar{\theta}_{2bc}(z), \\ &\Phi_{c1}(z) = (U_{bc}+1)\Phi_0(z) + \theta_{2bc}(z), \end{aligned}$$

where

$$\begin{aligned} &V_{bc} = \frac{G_c - G_b}{G_b + G_c\kappa_b}, \quad U_{bc} = \frac{G_c\kappa_b - G_b\kappa_c}{G_b\kappa_c + G_c}, \theta_{1bc}(z) = \frac{-2G_bG_c\beta_b}{G_b + G_c\kappa_b}\theta_{bc}(z), \\ &\theta_{2bc}(z) = \frac{2G_bG_c}{G_b\kappa_c + G_c}\{\beta_b[\theta_0(z) + \theta_{ba}(z)] - \beta_c\theta_c(z)\} \end{aligned} \tag{27}$$

and the homogeneous solutions are

$$\Phi_0(z) = \frac{G_bq_0\beta_b}{\pi k_b(\kappa_b+1)}\log\left(\frac{z-z_s}{z-z_k}\right), \quad \Omega_0(z) = \frac{G_bq_0\beta_b}{\pi k_b(\kappa_b+1)}\left[\log\left(\frac{z-z_s}{z-z_k}\right) + \frac{z-\bar{z}_s}{z-z_s} - \frac{z-\bar{z}_k}{z-z_k}\right]. \tag{28}$$

Since the result is based on the assumption that D_a is made up of material b, it cannot satisfy the continuity condition across L^*.

Next, we assume regions D_b and D_c are made up of the same material b, and region D_a is composed of material a. Additional terms, $\Phi_{b1}(z)$ and $\Omega_{b1}(z)$ (analytic in $D_b \cup D_c$) and $\Phi_{a1}(z)$ and $\Omega_{a1}(z)$ (analytic in D_a) are introduced to satisfy the continuity conditions across the interface L^*, so that

$$\begin{aligned} &\Phi(z) = \begin{cases} \Phi_{a1}(z), & z \in D_a \\ \Phi_1(z) + \Phi_{b1}(z), & z \in D_b \cup D_c \end{cases}, \\ &\Omega(z) = \begin{cases} \Omega_{a1}(z), & z \in D_a \\ \Omega_1(z) + \Omega_{b1}(z), & z \in D_b \cup D_c \end{cases}. \end{aligned} \tag{29}$$

From the continuity of traction and displacement across L^* and standard analytic continuation arguments, we find

$$\begin{aligned} &\Phi_{b1}(z) = V_{ba}[\bar{\Omega}_1(z-2ih) + 2ih\bar{\Phi}_1'(z-2ih)] + \theta_{1ba}(z), \\ &\Omega_{a1}(z) = (V_{ba}+1)[\Omega_1(z) - 2ih\Phi_1'(z)] + 2ih\Phi_{a1}'(z) + \bar{\theta}_{1ba}(z-2ih), \end{aligned} \tag{30a}$$

$$\Omega_{b1}(z) = U_{ba}\bar{\Phi}_1(z-2ih) + 2ih\Phi_{a1}'(z) + \bar{\theta}_{2ba}(z-2ih), \quad \Phi_{a1}(z) = (U_{ba}+1)\Phi_1(z) + \theta_{2ba}(z),$$

where

$$V_{ba} = \frac{G_a - G_b}{G_b + G_a\kappa_b}, \quad U_{ba} = \frac{G_a\kappa_b - G_b\kappa_a}{G_b\kappa_a + G_a},$$

$$\theta_{1ba}(z) = \frac{-2G_aG_b\beta_b}{G_b + G_a\kappa_b}\theta_{ba}(z), \quad \theta_{2ba}(z) = \frac{2G_aG_b}{G_b\kappa_a + G_a}\{\beta_b[\theta_0(z) + \theta_{bc}(z)] - \beta_a\theta_a(z)\}. \tag{30b}$$

The method of analytical continuation is repeatedly performed across the two interfaces to generate the additional terms. Since all these procedures are similar to the previous approach, the details are omitted here. The final results are as follows.

$$\Phi(z) = \begin{cases} \sum_{i=1}^{n} \Phi_{ai}(z), & z \in D_a \\ \Phi_0(z) + \sum_{i=1}^{n} \Phi_{bi}(z) + \sum_{i=1}^{n} \Phi_i(z), & z \in D_b \\ \sum_{i=1}^{n} \Phi_{ci}(z), & z \in D_c \end{cases},$$
$$\Omega(z) = \begin{cases} \sum_{i=1}^{n} \Omega_{ai}(z), & z \in D_a \\ \Omega_0(z) + \sum_{i=1}^{n} \Omega_{bi}(z) + \sum_{i=1}^{n} \Omega_i(z), & z \in D_b \\ \sum_{i=1}^{n} \Omega_{ci}(z), & z \in D_c \end{cases}. \tag{31}$$

These include the following summations.

$$\sum_{i=1}^{n} \Phi_{ai}(z) = \sum_{i=1}^{n} (U_{ba} + 1)\Phi_i(z) + \theta_{2ba}(z),$$
$$\sum_{i=1}^{n} \Phi_{bi}(z) = \sum_{i=1}^{n} \{V_{ba}[\bar{\Omega}_i(z - 2\mathrm{i}h) + 2\mathrm{i}h\bar{\Phi}_i'(z - 2\mathrm{i}h)]\} + \theta_{1ba}(z),$$
$$\sum_{i=1}^{n} \Phi_i(z) = \sum_{i=1}^{n} V_{bc}\bar{\Omega}_{i-1}(z) + \theta_{1bc}(z), \quad \sum_{i=1}^{n} \Phi_{ci}(z) = \sum_{i=1}^{n} (U_{bc} + 1)\Phi_{i-1}(z) + \theta_{2bc}(z),$$
$$\sum_{i=1}^{n} \Omega_{ai}(z) = \sum_{i=1}^{n} \{(V_{ba} + 1)[\Omega_i(z) - 2\mathrm{i}h\Phi_i'(z)] + 2\mathrm{i}h\Phi_{ai}'(z)\} + \bar{\theta}_{1ba}(z - 2\mathrm{i}h),$$
$$\sum_{i=1}^{n} \Omega_{bi}(z) = \sum_{i=1}^{n} \{U_{ba}\bar{\Phi}_i(z - 2\mathrm{i}h) + 2\mathrm{i}h\Phi_{ai}'(z)\} + \bar{\theta}_{2ba}(z - 2\mathrm{i}h),$$
$$\sum_{i=1}^{n} \Omega_i(z) = \sum_{i=1}^{n} U_{bc}\bar{\Phi}_{i-1}(z) + \bar{\theta}_{2bc}(z), \quad \sum_{i=1}^{n} \Omega_{ci}(z) = \sum_{i=1}^{n} (U_{bc} + 1)\Phi_{i-1}(z) + \bar{\theta}_{1bc}(z). \tag{32}$$

Note that the homogeneous solutions indicated in Eq. 28 are for the singularities located in region D_b. For the singularities located in other regions, the solution can also be found by using the same procedure.

4 Thermoviscoelastic formulation of a trimaterial

For a linear thermoviscoelastic material, the strains or stresses at any given time are the sum of the individual strain or stress increments through the respective time intervals during which they have been applied. By Boltzman's superposition principle, the relationship between strains and stresses can be written in the form of a hereditary integral [10]

$$\varepsilon_m(\tau) = \tilde{s}_{mn}(0)\sigma_n(\tau) + \int_0^\tau \tilde{s}'_{mn}(\tau-\xi)\sigma_n(\xi)\mathrm{d}\xi + \tilde{\alpha}_m(0)T(\tau) + \int_0^\tau \tilde{\alpha}'_m(\tau-\xi)T(\xi)\mathrm{d}\xi, \quad \text{for } m,n=1,2,\ldots,6. \tag{33}$$

Here, ξ is the dummy variable regarding the argument in question. Further $\tau = tb(T)$ is designated as the reduced time, and the function $b(T)$ indicates the temperature shift function which characterizes the time-dependent properties of the thermorhelogically simple material [10]. By using the Laplace transform, we obtain Eq. 33 as follows:

$$\hat{\varepsilon}_m(p) = \hat{s}_{mn}(p)\hat{\sigma}_n(p) + \hat{\alpha}_m(p)\hat{T}(p), \tag{34}$$

where $\hat{s}_{mn}(p) = p\hat{\tilde{s}}_{mn}(p)$, and $\hat{\alpha}_m = p\hat{\tilde{\alpha}}_m(p)$. Equation 34 is analogous to the thermoelastic constitutive equation. Consequently, similar to the thermoelastic problem in the previous discussion, the thermoviscoelastic field can be written as

$$2\hat{G}(\hat{u}'_1 + \hat{u}'_2) = \kappa\hat{\Phi}(\hat{z}) - \hat{\bar{\Omega}}(\hat{\bar{z}}) - (\hat{z} - \hat{\bar{z}})\,\hat{\bar{\Phi}}'(\hat{\bar{z}}) + 2\hat{G}\hat{\beta}\hat{\theta}(\hat{z}), \tag{35}$$

$$\hat{\sigma}_{22} - i\hat{\sigma}_{12} = \hat{\Phi}(\hat{z}) + \hat{\bar{\Omega}}(\hat{\bar{z}}) + (\hat{z} - \hat{\bar{z}})\,\hat{\bar{\Phi}}'(\hat{\bar{z}}), \tag{36}$$

where all coefficients in Eqs. 35 and 36 can be obtained by simple alternation from the previous thermoelastic definition. Then, the real-time solution can be found numerically by the direct inverse Laplace transform

$$\boldsymbol{\varphi}_i(z) = \frac{1}{2\pi\mathrm{i}}\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty} \hat{\boldsymbol{\varphi}}_i(\hat{z})\mathrm{e}^{p\tau}\mathrm{d}p = \boldsymbol{R}_0 + \sum_{i=1}^{m}\boldsymbol{R}_i\mathrm{e}^{-\omega_i\tau}, \tag{37}$$

where $\boldsymbol{R}_0$ is a real constant obtained from the boundary condition at $\tau \to \infty$ and the $\boldsymbol{R}_i$ terms are to be evaluated from the set of linear algebraic equations,

$$\left[\hat{\boldsymbol{\varphi}}_i(\hat{z}) - \frac{\boldsymbol{R}_0}{p}\right]_{p\to a_i} = \left[\sum_{i=1}^{m}\frac{\boldsymbol{R}_i}{p+\omega_i}\right]_{p\to a_i}, \tag{38}$$

where $-\omega_i$ denotes the poles of $\hat{\boldsymbol{\varphi}}(\hat{z})$ in the p-domain (except the origin), and a_i may be any arbitrary constant excluding the poles. Equation 38 represents a system of m linear algebraic equations to be used to determine the m unknown coefficients $\boldsymbol{R}_i$ in the assumed form of the solution in Eq. 37.

5 Numerical results

Some typical examples of the interfacial stresses of a trimaterial subjected to a pair of a point heat source and a sink (see Fig. 1) are shown in this section.

5.1 Interfacial stress problem—general case for a trimaterial

From Eq. 33, the general thermoviscoelastic constitutive equation can be expressed as

$$\varepsilon_m(\tau) = s^0_{mn}\left[\sigma_n(\tau) + \sum_{i=1}^{N}\int_0^\tau g_i(\tau-\xi)\sigma_n(\xi)\mathrm{d}\xi\right] + \alpha^0_m\left[T(\tau) + \sum_{i=1}^{N}\int_0^\tau g_{ti}(\tau-\xi)T(\xi)\mathrm{d}\xi\right], \tag{39}$$

where

$$g_i(\tau) = \eta_i\mathrm{e}^{-\tau/\lambda_i}, \quad g_{ti}(\tau) = \eta_{ti}\mathrm{e}^{-\tau/\lambda_{ti}}. \tag{40}$$

For a thermorheologically simple material associated with a Kelvin–Maxwell three-parameter model, Eq. 39 becomes

$$\varepsilon_m(\tau) = s^0_{mn}\left[\sigma_n(\tau) + \int_0^\tau g(\tau-\xi)\sigma_n(\xi)\mathrm{d}\xi\right] + \alpha^0_m\left[T(\tau) + \int_0^\tau g_t(\tau-\xi)T(\xi)\mathrm{d}\xi\right], \tag{41}$$

where the reduced time, $\tau = tb(T)$ is defined in Eq. 33. Also,

$$g(\tau) = \eta \mathrm{e}^{-\tau/\lambda}, \quad \eta = \frac{s^\infty_{mn} - s^0_{mn}}{\lambda s^0_{mn}}, \quad g_t(\tau) = \eta_t \mathrm{e}^{-\tau/\lambda_t}, \quad \eta_t = \frac{\alpha^\infty_m - \alpha^0_m}{\lambda_t \alpha^0_m}, \tag{42}$$

where λ and λ_t indicate the relaxation constants, and s^0_{mn}, s^∞_{mn}, α^0_m and α^∞_m denote the creep compliance and thermal expansion at $t = 0$ and ∞, respectively. For an isothermal uni-axial constant load σ_0, the thermoviscoelastic model in Eq. 41 indicates that $\varepsilon(\tau)$ approaches to $s^0_{mn}\sigma_0$ at $\tau = 0$, and converges to $s^\infty_{mn}\sigma_0$ when $\tau \to \infty$. It is also clear that $s^0_{mn} = s^\infty_{mn}$, $\alpha^0_m = \alpha^\infty_m$ and $\eta = \eta_t = 0$ for a thermoelastic material. Moreover, $\varepsilon(\tau)$ approaches $\varepsilon(\infty)$ rapidly as the relaxation constants λ and λ_t decrease. Taking the Laplace transform with respect to the reduced time τ, we may write Eq. 41 as follows:

$$\hat{\varepsilon}_m(p) = s^0_{mn}\left(1 + \frac{\eta}{p + 1/\lambda}\right)\hat{\sigma}_n(p) + \alpha^0_m\left(1 + \frac{\eta_t}{p + 1/\lambda_t}\right)\hat{T}(p). \tag{43}$$

For an isotropic material, the relations between elastic compliances and the modulus are described by

$$s^0_{mn} = \frac{1}{G}\begin{bmatrix} \frac{1}{2(1+\nu)} & \frac{-\nu}{2(1+\nu)} & \frac{-\nu}{2(1+\nu)} & 0 & 0 & 0 \\ & \frac{1}{2(1+\nu)} & \frac{-\nu}{2(1+\nu)} & 0 & 0 & 0 \\ & & \frac{1}{2(1+\nu)} & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & \text{sym.} & & & 1 & 0 \\ & & & & & 1 \end{bmatrix}, \tag{44}$$

where G and ν denote the elastic shear modulus and Poisson's ratio, respectively. Meanwhile, the thermal-expansion coefficients are

$$\alpha^0_m = \left[\alpha\ \alpha\ \alpha\ 0\ 0\ 0\right]^T. \tag{45}$$

In the following discussion, we assume the following values. For the elastic constants of material $a(D_a)$, we have,

$$G = G_a, \quad \nu = 0.3, \quad k = k_a, \quad \alpha = \alpha_a, \quad \lambda = \lambda_t, \quad \eta = \eta_t = 0, \quad b(T) = 1.$$

For the constants of material $b(D_b)$, we have,

$$G = G_b, \quad \nu = 0.3, \quad k = k_b, \quad \alpha = \alpha_b, \quad \lambda = \lambda_t, \quad \eta = \eta_t = 0.5/\lambda, \quad b(T) = b_1 \mathrm{e}^{b_2(T/T_0)}, \quad b_1 = 1, \quad b_2 = 1, \quad T_0 = 298,$$

where T_0 denotes the reference temperature. For the constants of material $c(D_c)$, we have,

$$G = G_c, \quad \nu = 0.3, \quad k = k_c, \quad \alpha = \alpha_c, \quad \lambda = \lambda_t, \quad \eta = \eta_t = 0, \quad b(T) = 1.$$

The material constants not mentioned here, except the symmetric terms, are assumed to be zero for an isotropic material. Note that, when $\eta = \eta_t = 0$ of the material $a(D_a)$ or $c(D_c)$, it indicates that the material properties of the medium are independent of time, or that the medium is assumed to be elastic for the sake of simplicity. In the following cases, let the point heat source be located at $z_s(x_{s1} = -h, x_{s2} = h/2)$, and the heat sink at $z_k(x_{k1} = h, x_{k2} = h/2)$.

5.2 Specific case 1—when $a(D_a)$ and $c(D_c)$ are the same material

Figure 2 shows the interfacial stresses along $L(x_2 = 0)$ at $t = 0$, with $G_a = G_c$, $k_a = k_c$, $\alpha_a = \alpha_c$, $G_c/G_b = 1.2$, $k_c/k_b = G_c/G_b$, and $\alpha_c/\alpha_b = G_b/G_c$. It indicates that the normal stress is skew-symmetric with respect to the

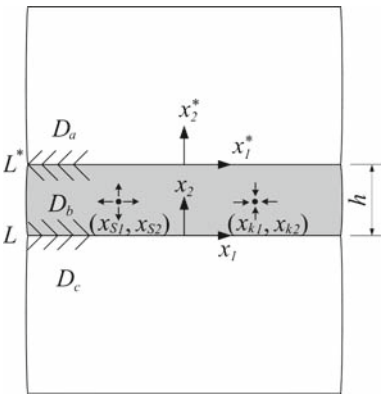

Fig. 1 A trimaterial with singularities in the middle layer

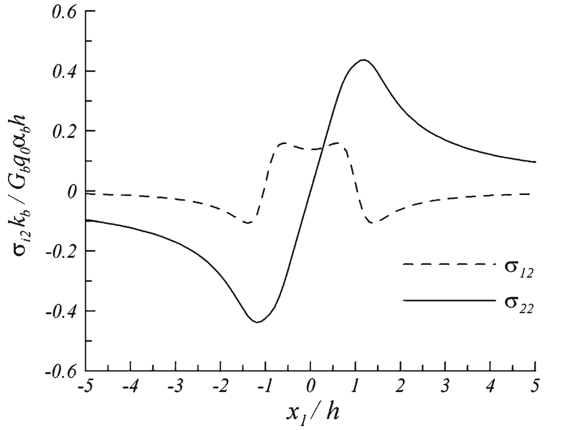

Fig. 2 Interfacial stresses of a triple-layer along $L(x_2 = 0)$ at $t = 0$, with $G_a = G_c$, $k_a = k_c$, $\alpha_a = \alpha_c$, $G_c/G_b = 1.2$, $k_c/k_b = G_c/G_b$, and $\alpha_c/\alpha_b = G_b/G_c$

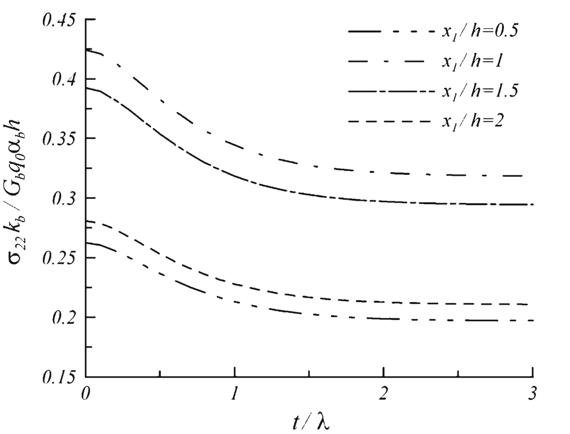

Fig. 3 Evolution of interfacial normal stresses of a triple-layer on $x_2 = 0$, with $G_a = G_c$, $k_a = k_c$, $\alpha_a = \alpha_c$, $G_c/G_b = 1.2$, $k_c/k_b = G_c/G_b$, and $\alpha_c/\alpha_b = G_b/G_c$

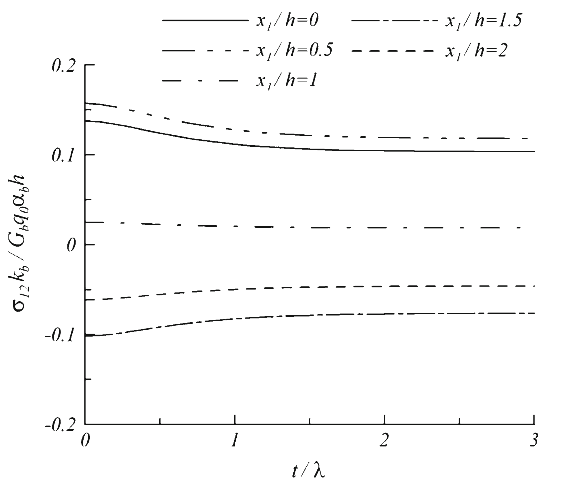

Fig. 4 Evolution of interfacial shear stresses of a triple-layer on $x_2 = 0$, with $G_a = G_c$, $k_a = k_c$, $\alpha_a = \alpha_c$, $G_c/G_b = 1.2$, $k_c/k_b = G_c/G_b$, and $\alpha_c/\alpha_b = G_b/G_c$

x_2-axis, which becomes positive at the heat-sink side and negative at the heat-source side. The maximum value of the normal stress occurs at $x_1/h = \pm 1.2$. Meanwhile, the shear stress is symmetric with respect to the x_2-axis and positive in the region between the heat source and sink. It also shows that the maximum shear stress occurs at $x_1/h = \pm 0.6$. Note that the positive or negative interfacial stresses depend on the choice of material properties for the trimaterial. The results agree with the thermoelastic solution of Chao and Chen [8]. The first five terms of the series solution are obtained in this work. In order to demonstrate the convergence of the series solution, the contribution of the stresses for the first five terms of a series solution is 73.35%, 18.68%, 4.030%, 0.837% and 0.103%, respectively. The contribution accounts for the ratio of each term to the summation of the first five terms of a series solution. The leading three terms have over 99% contribution, making the series solution rapidly convergent. Figure 3 shows the evolution of interfacial normal stresses at points $x_1/h = 0.5, 1, 1.5$, and $2 (x_2 = 0)$, respectively. It indicates the normal stresses decrease with time because of thermoviscoelastic effects of material b. Besides, the normal stress vanishes at the point $x_1/h = 0$. Figure 4 shows the evolution of interfacial shear stresses at points $x_1/h = 0, 0.5, 1, 1.5$, and $2(x_2 = 0)$. It indicates that the absolute value of the shear stress decreases with time.

5.3 Specific case 2—the thin-film problem

The trimaterial problem can be reduced to a finite thickness film/substrate problem by letting the upper or lower medium vanish. Figure 5 shows the interfacial stresses of a film/substrate, assuming $G_a \to 0$, $k_a \to 0$, and $G_c/G_b = 1.2$, $k_c/k_b = G_c/G_b$, $\alpha_c/\alpha_b = G_b/G_c$. It shows that the interfacial stresses for the film/substrate problem have the same tendency as those in the trimaterial problem (comparing to Fig. 2), except that the values of the film/substrate problem (Fig. 5) are larger. This happens because the free surface of the finite-thickness film/substrate (L^*) is insulated ($k_a = 0$), leaving the heat to flow to the substrate, resulting in an increase of the temperature on the interface L. The results also agree well with the thermoelastic solution of Chao and Chen [8]. Figure 6 shows the evolution of interfacial stresses of a film/substrate at $x_1/h = 0.5, 1, 1.5$, and 2. It shows the normal stresses decreasing with time at the beginning, and then converging to constants after a period of time. Figure 7 shows

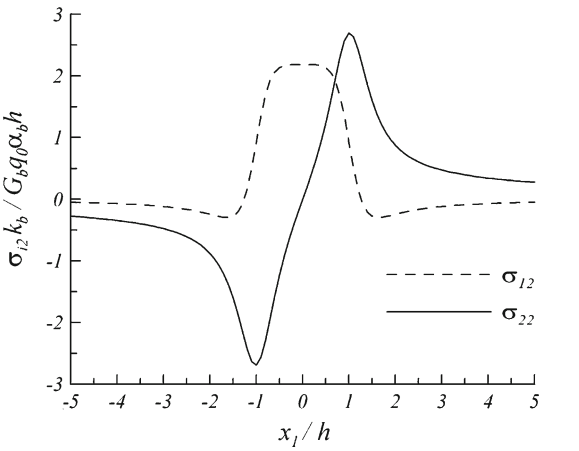

Fig. 5 Interfacial stresses of a film/substrate along $L(x_2 = 0)$ at $t = 0$, with $G_c/G_b = 1.2$, $k_c/k_b = G_c/G_b$, and $\alpha_c/\alpha_b = G_b/G_c$

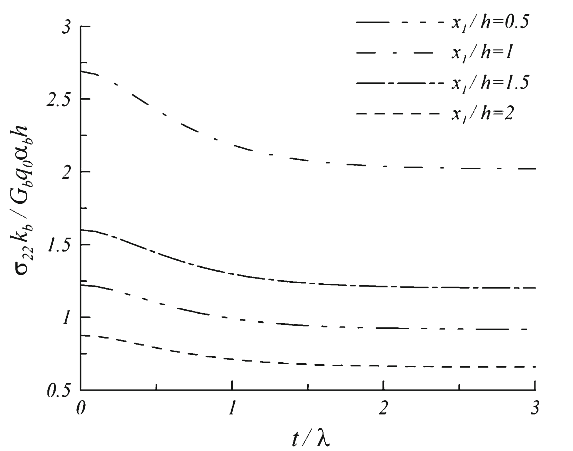

Fig. 6 Evolution of interfacial normal stresses of a film/substrate, with $G_c/G_b = 1.2$, $k_c/k_b = G_c/G_b$, and $\alpha_c/\alpha_b = G_b/G_c$

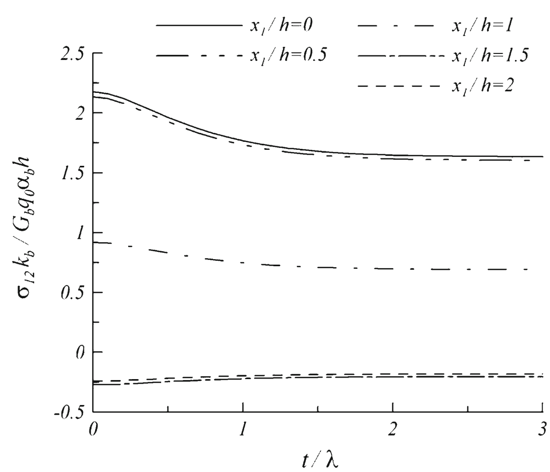

Fig. 7 Evolution of interfacial shear stresses of a film/substrate, with $G_c/G_b = 1.2$, $k_c/k_b = G_c/G_b$, and $\alpha_c/\alpha_b = G_b/G_c$

the interfacial shear stresses of a film/substrate, which also indicates that the absolute values of the shear stresses decrease with time because of the thermoviscoelastic effect of material b.

6 Conclusions

A thermoviscoelastic analysis of a dissimilar trimaterial has been presented. Using analytic continuation associated with the successive alternating technique, the solution can be found in terms of a rapidly convergent series. The rate of convergence of the thermal potential depends on the bimaterial constants Λ_{cb}, Π_{cb}, Λ_{ab}, and Π_{ab}, while the stress function depends on V_{cb}, U_{cb}, V_{ab}, and U_{ab}. The present series solutions converge to the true solution since those bimaterial constants are always less than one. The convergence rate becomes more rapid as the differences of the elastic constants of the neighboring materials get smaller. Even if materials a and/or c are rigid or non-existent, the solution remains valid. Moreover, the thermoviscoelastic analysis of a triple-layered medium, as well as that of a film/substrate, shows that the interfacial stresses induced by a pair of heat source and sink decrease with time.

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